1 Cauchy-Riemann equations

Let us consider the $x - y$ 2D Cartesian system of coordinates and define the complex functional $Z(z)$, with the complex variable $z \in \mathbf{C}$ defined as

$$z = x + i\,y,\ i = \sqrt{-1}. \tag{1}$$

The derivation of complex functional can be eased using Cauchy-Riemann equations:

$$\mathrm{Re}\frac{dZ}{dz} = \frac{\partial(\mathrm{Re}Z)}{\partial x} = \frac{\partial(\mathrm{Im}Z)}{\partial y}, \tag{2}$$

$$\mathrm{Im}\frac{dZ}{dz} = \frac{\partial(\mathrm{Im}Z)}{\partial x} = -\frac{\partial(\mathrm{Re}Z)}{\partial y}. \tag{3}$$

Those equations form a necessary and sufficient condition for a complex function of a complex variable to be complex differentiable. Show that for the function $Z(z) = z^2$, the Cauchy-Riemann equations are satisfied.

2 Westergaard's solution

In 1939, H.M. Westergaard combined complex analysis with Airy stress function to give the analytical stress field of several mechanical systems. After recalling the required concepts of complex analysis, this exercise will present one of Westergaard's solutions.
Let us define some useful notations used by Westergaard for derivatives:

$$\frac{dZ}{dz} = Z', \tag{4}$$

$$\frac{dZ'}{dz} = Z'', \tag{5}$$

as well as primitive functions of Z

$$\frac{d\overline{Z}}{dz} = Z, \tag{6}$$

$$\frac{d\overline{\overline{Z}}}{dz} = \overline{Z}. \tag{7}$$

Question 1

Within the aforementioned system of coordinates, Westergaard defined the Airy stress function $\phi = \mathrm{Re}\overline{\overline{Z}} + y\mathrm{Im}\overline{Z}$. Using the biharmonic equation, show that this function is an admissible Airy function whatever the definition of Z (use the Cauchy-Riemann equations).

Question 2

Express the components of the stress tensor $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ as function of Z and Z'.

Question 3

Following this formalism, Westergaard defined different complex functionals Z, each being the analytical solution of a mechanical problem. Let us focus on one of them,

$$Z = \frac{\sigma_\infty}{\sqrt{1 - (a/z)^2}}, \tag{8}$$

where σ_∞ and a are real scalars. Using this definition of Z, compute and draw the evolution of the stress tensor components along the axis $y = 0$. This tensor represents the exact stress field of a mechanical system, find which one.

3 Irwin's solution

In the previous exercise, we have studied Westergaard's complex stress functions and their ability to compute the exact stress field caused by a sharp crack in an infinite plate. In 1957, Irwin used exactly the same approach to propose an approximation of the elastic solution at the vicinity of the crack tip and define the concept of stress intensity factor. This exercise will follow precisely his approach. Let's start from Westergaard's solution for a sharp crack of size a submitted to a remote loading σ_∞:

$$\sigma_{xx} = \sigma_{yy} = \frac{\sigma_\infty}{\sqrt{1-(a/z)^2}}, \tag{9}$$

z being a complex variable defined in the $x - y$ Cartesian system of coordinates as $z = x + i\,y$.

Question 1

A first step consist in moving toward a polar system of coordinates $r - \theta$ centered at the position of the crack tip $x = a, y = 0$. Within this system of coordinates, rewrite Westergaard's solution along the line $y = 0$.

Question 2

As presented in the lecture, Irwin observed that the stress field of any crack tip can be approximated at its very vicinity by

$$\sigma_{ij}(r,\theta) \cong \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta), \tag{10}$$

where f_{ij} are universal function depending only on the angular coordinates as

$$\left\{ \begin{array}{l} f_{xx} = \cos(\frac{\theta}{2}) \left[1 - \sin(\frac{\theta}{2})\sin(\frac{3\theta}{2})\right], \\ f_{yy} = \cos(\frac{\theta}{2}) \left[1 + \sin(\frac{\theta}{2})\sin(\frac{3\theta}{2})\right]. \end{array} \right. \tag{11}$$

The pre-factor $\frac{1}{\sqrt{2\pi r}}$ characterizes the singularity of the stress field, i.e. how fast are the stresses diverging to infinity as we approach the crack tip. The resulting behavior is often referred as *the square root singularity* surrounding crack tips. Last but not least, a factor K, independent of both r and θ, called *the stress intensity factor* represents the effect of the geometry and loading conditions of the problem on the severity of stresses near the tip.
As example, let us compute the stress intensity factor of the infinite plate system. From Westergaard's solution written in polar coordinates (Question 1) express

$$K \cong \sigma_{ij}(r,\theta=0)\,\frac{\sqrt{2\pi r}}{f_{ij}(\theta=0)}. \tag{12}$$

Question 3

Irwin approximation is only valid at the very vicinity of the crack tip ($r \ll a$). Using limit analysis, compute the stress intensity factor of the infinite plate geometry

$$K = \lim_{r \ll a} \left(\sigma_{ij}(r,\theta=0)\,\frac{\sqrt{2\pi r}}{f_{ij}(\theta=0)} \right). \tag{13}$$

Using your favorite plotter, compare the exact stress field (Equation 9) with Irwin's approximation (Equation 10) along the line $y = 0$.